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Incorporation of anomalous magnetic moments in the two-body relativistic wave equations of constraint theory

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Abstract

Using a Dirac-matrix substitution rule, applied to the electric charge, the anomalous magnetic moments of fermions are incorporated in local form in the two-body relativistic wave equations of constraint theory. The structure of the resulting potential is entirely determined, up to magnetic type form factors, from that of the initial potential descibing the mutual interaction in the absence of anomalous magnetic moments. The wave equations are reduced to a single eigenvalue equation in the sectors of pseudoscalar and scalar states ($j = 0$). The requirement of a smooth introduction of the anomalous magnetic moments imposes restrictions on the behavior of the form factors near the origin, in x-space. The nonrelativistic limit of the eigenvalue equation is also studied.

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1 Introduction

The two-body relativistic wave equations of constraint theory have the main feature of describing the internal dynamics of the system by means of a manifestly covariant three-dimensional formalism [1, 2, 3, 4, 5, 6, 7, 8, 9]; relative energy and relative time variables are eliminated there through constraint equations. These wave equations, which can be constructed from general principles, have also the property of allowing a three-dimensional reduction of the Bethe-Salpeter equation by means of a Lippmann-Schwinger-quasipotential type equation [10, 11, 12, 13, 14, 15, 16, 17, 18] that relates the two-body potential to the scattering amplitude [19, 20]. In this way, the potential becomes calculable, in perturbation theory, from Feynman diagrams.

In a recent work [21], we applied this calculational method to the evaluation, in certain approximations, of the potentials in the cases of scalar and vector interactions, mediated by massless photons. It turns out that at each formal order of perturbation theory, which is now reorganized by the presence of additional three-dimensional diagrams due to the constraints, the leading infra-red terms are free of spurious singularities and can be represented in three-dimensional x-space as local functions of r, proportional to $(g^2/r)^n$, where r is the c.m. relative distance, g the coupling constant and n the formal order of perturbation theory. The series of leading terms can be summed and result in local functions (in r) for the expressions of the potentials. The latter are compatible with the potentials proposed by Todorov in the quasipotential approach on the basis of minimal substitution rules [16] and later investigated in the fermionic case by Crater and Van Alstine [7].

The purpose of the present paper is to take into account, in the case of vector interactions, the effects of the anomalous magnetic moments of fermions. In usual calculations in QED, because of the smallness of the coupling constant, the latter are evaluated at leading order of perturbation theory in the nonrelativistic limit ($O(\alpha^5)$ effect, where α is the fine structure constant). However, in strong coupling problems, like in the strong coupling regime of QED or in the short-distance (vector) interactions of QCD, nonperturbative contributions of the anomalous magnetic moments may become sizable and the incorporation of higher order effects becomes necessary.

In order to include the main effects of the anomalous magnetic moments into the potentials, we evaluate their contributions through the vertex corrections. In the lowest

order graph the anomalous magnetic moment appears by means of a substitution rule that replaces each charge (coupling constant) by a Dirac-matrix function. We then introduce this typical vertex correction at the vertices in each order of the perturbation series of the vector potential determined previously [21]. Although the substitution rule utilized above is rather simple, it generates in the higher order terms technical complications for the summation of the perturbation series of the potential. The reason for this is the new Dirac-matrix structure that results from the higher order terms. Up to now, all the potentials that were considered in the constraint theory wave equations had dependences on the Dirac matrices only through pairs of γ_1 and γ_2 matrices, the indices 1 and 2 referring to the two fermions, respectively (like $\gamma_1.\gamma_2$, $\gamma_{15}\gamma_{25}$, etc.)− a feature that considerably simplifies many algebraic operations as well as the reduction process to the final eigenvalue equation. The presence of the vertex corrections, even though globally symmetric in the exchanges $1 \leftrightarrow 2$, breaks this symmetry in the individual terms and introduces new types of structure not present in previous calculations. It is the presence of these terms that makes calculations rather complicated. While the potential can still be represented in a somehow compact form, the final eigenvalue equation for general quantum numbers becomes less easy to handle. It takes a relatively simple form only in the sectors of $j = 0$ states (pseudoscalar and scalar), to which we have limited our final analysis. The ground states of these sectors are precisely those which may be concerned with spontaneous breakdowns of symmetries (chiral and dilatational).

The plan of the paper is as follows. In Sec. 2, we consider lowest order perturbation theory and determine the substitution rule to be used for the vertex correction. In Sec. 3, the vertex corrections are incorporated into the vector potential. The new form of the latter is determined in Sec. 4, by resumming the corresponding perturbation series. In Sec. 5, the wave equations are reduced, for the $j = 0$ states, to a final eigenvalue equation. In Sec. 6, we analyze the effects of the anomalous magnetic moments in several limiting cases. The requirement from the accompanying form factors of not aggravating the singularities of the initial potential, leads to restrictions on their behavior near the origin in x-space. The nonrelativistic as well as the one-particle limits are also checked. Concluding remarks follow in Sec. 7.

2 Structure of the two-body potential with anomalous magnetic moments

In order to determine the structure of the potential in the presence of anomalous magnetic moments, we start from the expression of the coupling of a pointlike particle (fermion 1) with charge e_1 and anomalous magnetic moment κ_1 to an external electromagnetic potential A_μ and to its field strength tensor $F_{\mu\nu}$:

$$(\gamma_{1\mu}A^\mu + \frac{1}{2}\kappa_1\sigma_{\mu\nu}F^{\mu\nu}) \tag{2.1}$$

$[\sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu, \gamma_\nu]]$. In the case of a mutual interaction with another particle 2, expression (2.1) represents the lowest order perturbation theory result, where potential A_μ is itself expressed in terms of the photon propagator and its coupling to particle 2 (which we suppose for the moment without anomalous magnetic moment):

$$A_\mu \ = \ D_{\mu\nu}\gamma_2^\nu \ , \tag{2.2}$$

where $D_{\mu\nu}$ is the photon propagator including the coupling constants at its two ends.

In the three-dimensional formalism of constraint theory, the Bethe-Salpeter kernel is projected on the constraint hypersurface and the wave function expanded around it [19, 20]. In the c.m. frame, this amounts to projecting the kernel on the hypersurface where the temporal component of the momentum transfer is zero. In a covariant formalism, one first decomposes four-vectors along transverse and longitudinal components with respect to the total momentum P:

$$\begin{aligned}
&P \ = \ p_1 + p_2 \ , \quad p \ = \ \frac{1}{2}(p_1 - p_2) \ , \quad X \ = \ \frac{1}{2}(x_1 + x_2) \ , \quad x \ = \ x_1 - x_2 \ , \\
&x_\mu^T \ = \ x_\mu - \hat{P}.x\hat{P}_\mu \ , \quad x_L \ = \ \hat{P}.x \ , \quad P_L \ = \ \sqrt{P^2} \ , \quad \hat{P}_\mu \ = \ \frac{P_\mu}{P_L} \ , \\
&x^{T2} \ = \ x^2 - (\hat{P}.x)^2 \ , \quad r \ = \ \sqrt{-x^{T2}} \ , \\
&\gamma_\mu^T \ = \ \gamma_\mu - \hat{P}.\gamma\hat{P}_\mu \ , \quad \gamma_L \ = \ \hat{P}.\gamma \ , \quad M \ = \ m_1 + m_2 \ .
\end{aligned} \tag{2.3}$$

Thus, in constraint theory, the propagator in Eq. (2.2) depends on x, in x-space, through x^T only; in the Feynman gauge, to which we stick throughout this work, it has the expression (in lowest order) [21]:

$$D_{\mu\nu} \ = \ g_{\mu\nu}D(x^{T2}, P_L) \ , \quad D(x^{T2}, P_L) \ = \ \frac{e_1e_2}{4\pi}\frac{1}{2P_Lr} \ . \tag{2.4}$$

Using Eqs. (2.2) and (2.4) the field strength tensor takes the form:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \;=\; 2\dot{D}(x^T_\mu \gamma_{2\nu} - x^T_\nu \gamma_{2\mu}) \\ = \frac{1}{r^2} D(x^T_\mu \gamma_{2\nu} - x^T_\nu \gamma_{2\mu}) \;, \tag{2.5}$$

where the dot operation represents derivation with respect to x^{T2}:

$$\dot{f} \;\equiv\; \frac{\partial}{\partial x^{T2}} f \;. \tag{2.6}$$

Expression (2.1) then becomes:

$$\gamma_{1\mu} A^\mu + \frac{1}{2}\kappa_1 \sigma_{1\mu\nu} F^{\mu\nu} = D[\gamma_1.\gamma_2 - i\frac{\kappa_1}{2r^2}(\gamma_1^T.x^T \gamma_1.\gamma_2 - \gamma_1.\gamma_2 \gamma_1^T.x^T)] \\ = \frac{1}{2} D(1 - i\frac{\kappa_1}{r}\gamma_1^T.\frac{x^T}{r})\gamma_1.\gamma_2 \\ + \gamma_{10}\gamma_{20}\Big[\frac{1}{2} D(1 - i\frac{\kappa_1}{r}\gamma_1^T.\frac{x^T}{r})\gamma_1.\gamma_2\Big]^\dagger \gamma_{10}\gamma_{20} \;. \tag{2.7}$$

[The dagger represents hermitian conjugation.] We deduce that to this order the anomalous magnetic moment appears through the following matrix substitution of the charge e_1:

$$e_1 \;\rightarrow\; e_1' = e_1(1 - i\frac{\kappa_1}{r}\gamma_1^T.\frac{x^T}{r}) \;. \tag{2.8}$$

The above calculations can be repeated at the particle 2 (antifermion) vertex, where the charge substitution becomes:

$$e_2 \;\rightarrow\; e_2' = e_2(1 - i\frac{\kappa_2}{r}\gamma_2^T.\frac{x^T}{r}) \;. \tag{2.9}$$

[e_2 is the fermion 2 charge; the passage to the antifermion is obtained in momentum space with the replacement $p_1 \rightarrow -p_2$, which yields for the momentum transfer $q = p_1 - p_1' \rightarrow -p_2 + p_2' = q$ and hence in x-space $x \rightarrow x$; the Dirac matrices γ_2 act on the wave function from the right.]

The mutual interaction potential then becomes to lowest order:

$$V = \frac{1}{2} D(1 - i\frac{\kappa_1}{r}\gamma_1^T.\frac{x^T}{r})(1 - i\frac{\kappa_2}{r}\gamma_2^T.\frac{x^T}{r})\gamma_1.\gamma_2 \\ + \gamma_{10}\gamma_{20}\Big[\frac{1}{2} D(1 - i\frac{\kappa_1}{r}\gamma_1^T.\frac{x^T}{r})(1 - i\frac{\kappa_2}{r}\gamma_2^T.\frac{x^T}{r})\gamma_1.\gamma_2\Big]^\dagger \gamma_{10}\gamma_{20} \;. \tag{2.10}$$

It is natural to generalize the substitution rules (2.8)-(2.9) to higher orders, with the difference that the lowest order anomalous magnetic term κ/r should now be replaced by

a form factor $b(r)$:

$$e_a \ \rightarrow \ e'_a = e_a\Big(1 - ib(r)\gamma_a^T.\frac{x^T}{r}\Big) \qquad (a = 1, 2) \ . \tag{2.11}$$

The form factor $b(r)$ should ensure for the anomalous magnetic term a smooth behavior at the origin, in order not to enhance the existing singularities of the propagator D. Taking into account the lowest order expression of κ, $\kappa = \frac{\hbar}{2m}\frac{\alpha}{2\pi}$, a convenient parametrization for $b(r)$ is:

$$\begin{aligned} b_a(r) &= \frac{\hbar\alpha/(2\pi)}{2mr + (\hbar\alpha/(2\pi))c_a(r)} \ , \qquad c_a(r) > 1 \qquad (a = 1, 2) \ , \\ \alpha &= \frac{e^2}{4\pi} \ , \qquad e_1 = e_2 = e \ . \end{aligned} \tag{2.12}$$

Expression (2.10) is then generalized to the following form:

$$\begin{aligned} V = \frac{1}{2}\Big(&A - iB_1\gamma_1^T.\frac{x^T}{r} - iB_2\gamma_2^T.\frac{x^T}{r} - C\gamma_1^T.\frac{x^T}{r}\gamma_2^T.\frac{x^T}{r}\Big)\gamma_1.\gamma_2 \\ &+ \frac{1}{2}\gamma_1.\gamma_2\Big(A + iB_1\gamma_1^T.\frac{x^T}{r} + iB_2\gamma_2^T.\frac{x^T}{r} - C\gamma_1^T.\frac{x^T}{r}\gamma_2^T.\frac{x^T}{r}\Big) \ , \end{aligned} \tag{2.13}$$

where the potentials A, B_1, B_2 and C are completely determined, by means of the substitution rules (2.11), from the expression of V in the absence of anomalous magnetic moments. The latter expression has the form:

$$V_0 \ = \ A_0\gamma_1.\gamma_2 \ . \tag{2.14}$$

[A_0 is denoted by V_2 in Ref. [21].]

For the Todorov potential [16, 7, 21], A_0 is:

$$A_0 \ = \ \frac{1}{4}\ln(1 + \frac{2\alpha}{P_L r}) \ . \tag{2.15}$$

However, other effective expressions could be used for A_0 as well.

Actually, the potentials that appear in the constraint theory wave equations are functions of V [Eq. (2.13)] through exponentiations; therefore, we shall need to calculate such exponential functions. This is the main content of Sec. 3.

3 Wave equations

The wave equations of constraint theory for a fermion-antifermion system can be written in the form [8, 22]:

$$
\begin{aligned}
(\gamma_1.p_1 - m_1)\ \widetilde{\Psi} &= (-\gamma_2.p_2 + m_2)\widetilde{V}\ \widetilde{\Psi}\ , \\
(-\gamma_2.p_2 - m_2)\ \widetilde{\Psi} &= (\gamma_1.p_1 + m_1)\widetilde{V}\ \widetilde{\Psi}\ ,
\end{aligned}
\tag{3.1}
$$

where $\widetilde{\Psi}$ is a spinor function of rank two, represented as a 4×4 matrix function; the Dirac matrices (γ_2) of the antifermion act on $\widetilde{\Psi}$ from the right; the total and relative variables were defined in Eqs. (2.3); $\widetilde{V}$ is a Poincaré invariant mutual interaction potential.

Equations (3.1) imply the constraint

$$
\Big[(p_1^2 - p_2^2) - (m_1^2 - m_2^2)\Big]\ \widetilde{\Psi}\ =\ 0\ , \tag{3.2}
$$

or equivalently

$$
C(p)\ \equiv\ 2P_Lp_L - (m_1^2 - m_2^2)\ \simeq\ 0\ , \tag{3.3}
$$

which allows the elimination from the wave equations of the relative longitudinal momentum in terms of the masses and the c.m. total energy. The wave function $\widetilde{\Psi}$, for eigenfunctions of the total momentum P, has then the structure:

$$
\widetilde{\Psi}(X,x)\ =\ e^{-iP.X}e^{-i(m_1^2 - m_2^2)x_L/(2P_L)}\ \widetilde{\psi}(x^T)\ . \tag{3.4}
$$

The positivity conditions of the norm of $\widetilde{\Psi}$ imply that $\widetilde{V}$ should satisfy the inequality $\frac{1}{4}Tr\widetilde{V}^\dagger\widetilde{V} < 1$ [8, 22]. A convenient parametrization satisfying this inequality for potentials commuting with $\gamma_{1L}\gamma_{2L}$ was proposed by Crater and Van Alstine [23]; it is: $\widetilde{V} = \tanh V$. It turns out that the perturbation series of the leading infra-red terms in QED in the Feynman gauge provides a potential V that is compatible with this parametrization [21] (cf. Eqs. (2.14)-(2.15), where V is denoted by V_0). For more general potentials which do not commute with $\gamma_{1L}\gamma_{2L}$, the generalization of the above parametrization is [22]:

$$
\gamma_{1L}\gamma_{2L}\widetilde{V}\ =\ \tanh(\gamma_{1L}\gamma_{2L}V)\ . \tag{3.5}
$$

Equations (3.1) are then transformed with the change of wave function:

$$
\widetilde{\Psi}\ =\ \cosh(\gamma_{1L}\gamma_{2L}V)\ \Psi\ ; \tag{3.6}
$$

they become:

$$(\gamma_1.p_1 - m_1)\cosh(\gamma_{1L}\gamma_{2L}V)\ \Psi = (-\gamma_2.p_2 + m_2)\gamma_{1L}\gamma_{2L}\sinh(\gamma_{1L}\gamma_{2L}V)\ \Psi\ ,$$
$$(-\gamma_2.p_2 - m_2)\cosh(\gamma_{1L}\gamma_{2L}V)\ \Psi = (\gamma_1.p_1 + m_1)\gamma_{1L}\gamma_{2L}\sinh(\gamma_{1L}\gamma_{2L}V)\ \Psi\ . \qquad (3.7)$$

One can also equivalently work in the "Breit representation". Defining

$$V_B = \gamma_{1L}\gamma_{2L}V\ , \qquad (3.8)$$
$$\Psi_B = e^{-V_B}\Psi\ , \qquad (3.9)$$

one shows that Eqs. (3.1) or (3.7) reduce to thye Breit type equation [22]:

$$\Big[P_L e^{2V_B} - (\mathcal{H}_1 + \mathcal{H}_2)\Big]\ \Psi\ =\ 0\ , \qquad (3.10)$$

provided constraint (3.2)-(3.3) is used; here, $\mathcal{H}_1$ and $\mathcal{H}_2$ are the covariant free hamiltonians:

$$\mathcal{H}_1 = m_1\gamma_{1L} - \gamma_{1L}\gamma_1^T.p_1^T\ ,$$
$$\mathcal{H}_2 = -m_2\gamma_{2L} - \gamma_{2L}\gamma_2^T.p_2^T\ . \qquad (3.11)$$

The normalization conditions of the wave functions $\widetilde{\Psi}$, Ψ and Ψ_B were presented in Ref. [22].

To solve the wave equations one decomposes the sixteen-component (4×4) wave function ψ along four-component (2×2) wave functions:

$$\psi\ =\ \psi_1 + \gamma_L\psi_2 + \gamma_5\psi_3 + \gamma_L\gamma_5\psi_4\ , \qquad (3.12)$$

and similarly for the Breit type wave function ψ_B:

$$\psi_B\ =\ \psi_{B1} + \gamma_L\psi_{B2} + \gamma_5\psi_{B3} + \gamma_L\gamma_5\psi_{B4}\ . \qquad (3.13)$$

These components are obtained with the projectors [22]

$$\mathcal{P}_1\ =\ \frac{1}{4}(1+\gamma_{1L}\gamma_{2L})(1+\gamma_{15}\gamma_{25})\ , \qquad \mathcal{P}_2\ =\ \frac{1}{4}(1+\gamma_{1L}\gamma_{2L})(1-\gamma_{15}\gamma_{25})\ ,$$
$$\mathcal{P}_3\ =\ \frac{1}{4}(1-\gamma_{1L}\gamma_{2L})(1+\gamma_{15}\gamma_{25})\ , \qquad \mathcal{P}_4\ =\ \frac{1}{4}(1-\gamma_{1L}\gamma_{2L})(1-\gamma_{15}\gamma_{25})\ . \qquad (3.14)$$

The spin operators, which act in the four-component wave function subspaces, are defined by means of the Pauli-Lubanski operators:

$$\begin{aligned}
W_{1S\alpha} &= -\frac{\hbar}{4}\epsilon_{\alpha\beta\mu\nu}P^\beta\sigma_1^{\mu\nu} \ , \quad W_{2S\alpha} \ = \ -\frac{1}{4}\epsilon_{\alpha\beta\mu\nu}P^\beta\sigma_2^{\mu\nu} \qquad (\epsilon_{0123}=+1) \ , \\
\gamma_{1L}W_{1S\alpha} &= \frac{\hbar P_L}{2}\gamma_{1\alpha}^T\gamma_{15} \ , \quad \gamma_{2L}W_{2S\alpha} \ = \ \frac{\hbar P_L}{2}\gamma_{2\alpha}^T\gamma_{25} \ , \\
W_{1S}^2 &= W_{2S}^2 \ = \ -\frac{3}{4}\hbar^2P^2 \ , \quad W_S \ = \ W_{1S}+W_{2S} \ , \\
w &\equiv \left(\frac{2}{\hbar P_L}\right)^2 W_{1S}.W_{2S} \underset{\text{c.m.}}{\longrightarrow} -\frac{4}{\hbar^2}\mathbf{s}_1.\mathbf{s}_2 \ , \\
w_{12} &\equiv \left(\frac{2}{\hbar P_L}\right)^2 \frac{W_{1S}.x^TW_{2S}.x^T}{x^{T2}} \underset{\text{c.m.}}{\longrightarrow} -\frac{4}{\hbar^2}\frac{(\mathbf{s}_1.\mathbf{x})(\mathbf{s}_2.\mathbf{x})}{\mathbf{x}^2} \ , \\
w_{12}^2 &= 1 \ , \quad w_{12}(w-w_{12}) \ = \ w-w_{12} \ .
\end{aligned} \tag{3.15}$$

It is clear, from Eqs. (3.7) and (3.10), that one has to calculate the exponential of $\gamma_{1L}\gamma_{2L}V$, with V having the general structure (2.13). We have:

$$\begin{aligned}
V_B &= \gamma_{1L}\gamma_{2L}V \\
&= \frac{1}{2}\Big(A+iB_1\gamma_1^T.\frac{x^T}{r}+iB_2\gamma_2^T.\frac{x^T}{r}-C\gamma_1.\frac{x^T}{r}\gamma_2.\frac{x^T}{r}\Big)\ \gamma_{1L}\gamma_{2L}\gamma_1.\gamma_2 \\
&\quad + \frac{1}{2}\gamma_{1L}\gamma_{2L}\gamma_1.\gamma_2\ \Big(A+iB_1\gamma_1^T.\frac{x^T}{r}+iB_2\gamma_2^T.\frac{x^T}{r}-C\gamma_1^T.\frac{x^T}{r}\gamma_2^T.\frac{x^T}{r}\Big) \ .
\end{aligned} \tag{3.16}$$

The difficulty of the calculation stems from the fact that the matrices $\gamma_1^T.x^T$ and $\gamma_2^T.x^T$ do not commute with $\gamma_1.\gamma_2$.

To proceed further, we introduce in the subspace of x^T a longitudinal direction, parallel to x^T, and a transverse plane, orthogonal to it. We define:

$$\begin{aligned}
\hat{x}_\mu^T &= \frac{x_\mu^T}{\sqrt{-x^{T2}}} \ = \ \frac{x_\mu^T}{r} \ , \quad \hat{x}^{T2} \ = \ -1 \ , \\
\gamma_{a\mu}^T &= \gamma_{a\mu}^\ell+\gamma_{a\mu}^t \ = \ -\gamma_a.\hat{x}^T\hat{x}_\mu^T+\gamma_{a\mu}^t \ , \quad \gamma_a^t.\hat{x}^T \ = \ 0 \ , \\
\gamma_{a\ell} &= \gamma_a^T.\hat{x}^T \ , \quad \gamma_{a\ell}^2 \ = \ -1 \qquad (a=1,2) \ .
\end{aligned} \tag{3.17}$$

The capital indices L and T [Eqs. (2.3)] concern the longitudinal and transverse components with respect to the total momentum P, while the small indices ℓ and t concern those of the three-dimensional relative distance x^T. We list here some useful relations satisfied by these matrices:

$$\begin{aligned}
\gamma_1.\gamma_2 &= \gamma_{1L}\gamma_{2L}+\gamma_1^T.\gamma_2^T \ = \ \gamma_{1L}\gamma_{2L}-\gamma_{1\ell}\gamma_{2\ell}+\gamma_1^t.\gamma_2^t \ , \\
\gamma_{1\ell}\gamma_{2\ell} &= -\gamma_{1L}\gamma_{2L}\gamma_{15}\gamma_{25}w_{12} \ , \quad \gamma_1^T.\gamma_2^T \ = \ -\gamma_{1L}\gamma_{2L}\gamma_{15}\gamma_{25}w \ ,
\end{aligned}$$

$$
\begin{aligned}
\gamma_1^t.\gamma_2^t &= -\gamma_{1\ell}\gamma_{2\ell}(w-w_{12}) \ , \qquad \Big[\gamma_{a\ell},\gamma_{aL}\Big]_+ = 0 \ , \\
\Big[\gamma_{a\ell},(w-w_{12})\Big]_+ &= 0 \ , \qquad \Big[\gamma_{a\ell},\gamma_{aL}(w-w_{12})\Big] = 0 \quad (a=1,2) \ . \qquad (3.18)
\end{aligned}
$$

($[\ ,\ ]_+$ is the anticommutator.)

At the first stage of the calculation, one can eliminate γ_{a5} and γ_a^t $(a=1,2)$ in terms of $\gamma_{a\ell}$, γ_{aL}, w and w_{12}. For the subspace of the matrices $\gamma_{a\ell}$, one introduces the following projectors:

$$
\begin{aligned}
\mathcal{P}_{++} &= \frac{1}{4}(1+i\gamma_{1\ell})(1+i\gamma_{2\ell}) \ , \qquad \mathcal{P}_{+-} = \frac{1}{4}(1+i\gamma_{1\ell})(1-i\gamma_{2\ell}) \ , \\
\mathcal{P}_{-+} &= \frac{1}{4}(1-i\gamma_{1\ell})(1+i\gamma_{2\ell}) \ , \qquad \mathcal{P}_{--} = \frac{1}{4}(1-i\gamma_{1\ell})(1-i\gamma_{2\ell}) \ , \qquad (3.19)
\end{aligned}
$$

which allows the decomposition of the $\gamma_{a\ell}$'s along the latter. Potential V_B [Eq. (3.16)] takes now the form:

$$
\begin{aligned}
V_B &= \gamma_{1L}\gamma_{2L}V = -(A\gamma_{1\ell}\gamma_{2\ell}-C)\gamma_{1L}\gamma_{2L} + (A - C\gamma_{1\ell}\gamma_{2\ell})(1+\gamma_{1L}\gamma_{2L}\gamma_1^t.\gamma_2^t) \\
&\qquad + i(B_1\gamma_{1\ell}+B_2\gamma_{2\ell})(1+\gamma_{1L}\gamma_{2L}\gamma_1^t.\gamma_2^t) \ . \qquad (3.20)
\end{aligned}
$$

Notice that the second term in the right-hand side above commutes with the two others and therefore its exponential can be factorized and calculated independently. The first and third terms can be written in terms of the projectors (3.19):

$$
\begin{aligned}
&-A\gamma_{1\ell}\gamma_{2\ell}\gamma_{1L}\gamma_{2L} + C\gamma_{1L}\gamma_{2L} + i(B_1\gamma_{1\ell}+B_2\gamma_{2\ell})(1+\gamma_{1L}\gamma_{2L}\gamma_1^t.\gamma_2^t) \\
&\quad = \mathcal{P}_{++}\Big[(A+C)\gamma_{1L}\gamma_{2L} + (B_1+B_2)(1+\gamma_{1L}\gamma_{2L}(w-w_{12}))\Big] \\
&\qquad + \mathcal{P}_{+-}\Big[(-A+C)\gamma_{1L}\gamma_{2L} + (B_1-B_2)(1-\gamma_{1L}\gamma_{2L}(w-w_{12}))\Big] \\
&\qquad + \mathcal{P}_{-+}\Big[(-A+C)\gamma_{1L}\gamma_{2L} - (B_1-B_2)(1-\gamma_{1L}\gamma_{2L}(w-w_{12}))\Big] \\
&\qquad + \mathcal{P}_{--}\Big[(A+C)\gamma_{1L}\gamma_{2L} - (B_1+B_2)(1+\gamma_{1L}\gamma_{2L}(w-w_{12}))\Big] \ . \qquad (3.21)
\end{aligned}
$$

The exponential of this expression can be calculated by a series expansion. The projectors $\mathcal{P}_{++}$, etc., commute with $\gamma_{1L}\gamma_{2L}(w-w_{12})$, but satisfy particular commutation rules with $\gamma_{1L}\gamma_{2L}$. One factorizes the projectors $\mathcal{P}_{++}$, etc., on the left of the series. Each multiplicative factor of $\mathcal{P}_{++}$, etc., can be resummed into exponential functions. At the end, one rexpresses $\gamma_{1\ell}$ and $\gamma_{2\ell}$ in terms of γ_{1L}, γ_{2L}, γ_{15}, γ_{25} and the spin operators (3.15) and one introduces back the projectors $\mathcal{P}_i$ $(i=1,\ldots,4)$, (3.14). One thus obtains:

$$
\begin{aligned}
e^{2V_B} &= e^{2\gamma_{1L}\gamma_{2L}V} \\
&= \frac{1}{2}(1+w_{12})e^{\alpha_+\gamma_+}(f_{++}+g_{++})\mathcal{P}_1 + \frac{1}{2}(1-w_{12})e^{\alpha_-\gamma_+}(f_{--}-g_{--})\mathcal{P}_1 \\
&\quad + \frac{1}{2}(1-w_{12})e^{\alpha_+\gamma_-}(f_{++}+g_{++})\mathcal{P}_2 + \frac{1}{2}(1+w_{12})e^{\alpha_-\gamma_-}(f_{--}-g_{--})\mathcal{P}_2 \\
&\quad + \frac{1}{2}(1-w_{12})e^{\alpha_+\gamma_+}(f_{+-}-g_{+-})\mathcal{P}_3 + \frac{1}{2}(1+w_{12})e^{\alpha_-\gamma_+}(f_{-+}+g_{-+})\mathcal{P}_3 \\
&\quad + \frac{1}{2}(1+w_{12})e^{\alpha_+\gamma_-}(f_{+-}-g_{+-})\mathcal{P}_4 + \frac{1}{2}(1-w_{12})e^{\alpha_-\gamma_-}(f_{-+}+g_{-+})\mathcal{P}_4 \\
&\quad + \frac{i}{2}(\gamma_{1\ell}+\gamma_{2\ell})\Big[e^{\alpha_+\gamma_+}h_{++}\mathcal{P}_1 + e^{\alpha_+\gamma_-}h_{++}\mathcal{P}_2 + e^{\alpha_+\gamma_+}h_{+-}\mathcal{P}_3 + e^{\alpha_+\gamma_-}h_{+-}\mathcal{P}_4\Big] \\
&\quad + \frac{i}{2}(\gamma_{1\ell}-\gamma_{2\ell})\Big[e^{\alpha_-\gamma_+}h_{--}\mathcal{P}_1 + e^{\alpha_-\gamma_-}h_{--}\mathcal{P}_2 + e^{\alpha_-\gamma_+}h_{-+}\mathcal{P}_3 + e^{\alpha_-\gamma_-}h_{-+}\mathcal{P}_4\Big] \ .
\end{aligned}
\tag{3.22}
$$

The definitions of the potential functions are the following:

$$
\begin{aligned}
f_{rs} &= \cosh\sqrt{\alpha_r^2+\beta_r^2\gamma_s^2}\ , \qquad g_{rs} = \frac{\alpha_r}{\sqrt{\alpha_r^2+\beta_r^2\gamma_s^2}}\sinh\sqrt{\alpha_r^2+\beta_r^2\gamma_s^2}\ , \\
h_{rs} &= \frac{\beta_r\gamma_s}{\sqrt{\alpha_r^2+\beta_r^2\gamma_s^2}}\sinh\sqrt{\alpha_r^2+\beta_r^2\gamma_s^2}\ , \qquad r,s=\pm\ , \\
f_{rs}^2 - g_{rs}^2 - h_{rs}^2 &= 1\ ,
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
\alpha_\pm &= 2(A\pm C)\ , \qquad \beta_\pm = 2(B_1\pm B_2)\ , \\
\gamma_\pm &= 1\pm(w-w_{12})\ .
\end{aligned}
\tag{3.24}
$$

The exponential function e^{-2V_B} is obtained from Eq. (3.22) by the replacements $\alpha\to-\alpha$ and $\beta\to-\beta$; e^{V_B} is obtained by the replacements $\alpha\to\alpha/2$ and $\beta\to\beta/2$, etc.. Also notice the commutation relations:

$$
\begin{aligned}
\gamma_{a\ell}\gamma_\pm &= \gamma_\mp\gamma_{a\ell}\ , \qquad \gamma_{a\ell}f_{\pm\pm} = f_{\pm\mp}\gamma_{a\ell}\ , \\
\gamma_{a\ell}g_{\pm\pm} &= g_{\pm\mp}\gamma_{a\ell}\ , \qquad \gamma_{a\ell}h_{\pm\pm} = h_{\pm\mp}\gamma_{a\ell}\ , \qquad a=1,2\ .
\end{aligned}
\tag{3.25}
$$

Equation (3.22) and the similar ones with different arguments allow us to project the wave equations (3.7) or (3.10) with the aid of the projectors $\mathcal{P}_i$ ($i=1,\ldots,4$), (3.14), appearing on the utmost right of the expressions, on the four-component wave functions (3.12) or (3.13). One thus obtains coupled equations for the four components ψ_i or ψ_{Bi} ($i=1,\ldots,4$) and eliminating three of them one reaches a final eigenvalue equation involving only one of the components.

4 Determination of the potentials

Before proceeding to the reduction of the wave equations, we shall determine the expressions of the various potentials appearing in Eqs. (3.16) and (3.22)-(3.24) in terms of the elementary Coulomb potential.

Potential V has the structure (2.13) and is obtained from Eq. (2.14) with the substitutions (2.11). We shall, for the moment, not use the particular expression (2.15) of the initial potential A_0, but rather present the calculations for the general case. We assume that A_0 is expressible as a power series of the elementary Coulomb potential $\alpha/(2P_Lr)$:

$$A_0 \ = \ A_0(v) \ = \ \sum_{n=1}^{\infty} a_n v^n \ , \tag{4.1}$$

$$v \ = \ \frac{\alpha}{2P_L r} \ . \tag{4.2}$$

The substitutions (2.11) yield for the fine structure constant α the modification:

$$\alpha \ \longrightarrow \ \alpha' = \alpha(1 - ib_1\gamma_{1\ell})(1 - ib_2\gamma_{2\ell}) \ , \tag{4.3}$$

which can be expressed in terms of the projectors (3.19):

$$\begin{aligned} \alpha' = \ & \alpha\Big[(1-b_1)(1-b_2)\mathcal{P}_{++} + (1-b_1)(1+b_2)\mathcal{P}_{+-} \\ & +(1+b_1)(1-b_2)\mathcal{P}_{-+} + (1+b_1)(1+b_2)\mathcal{P}_{--}\Big] \ . \end{aligned} \tag{4.4}$$

Since v [Eq. (4.2)] is proportional to α, its modification is similar to that given by Eq. (4.4). The initial potential A_0 thus undergoes the change:

$$\begin{aligned} A_0 \ \to \ \overline{A}_0 = \ & \sum_{n=1}^{\infty} a_n v^n \Big[\mathcal{P}_{++}(1-b_1)^n(1-b_2)^n + \mathcal{P}_{+-}(1-b_1)^n(1+b_2)^n \\ & +\mathcal{P}_{-+}(1+b_1)^n(1-b_2)^n + \mathcal{P}_{--}(1+b_1)^n(1+b_2)^n\Big] \\ = \ & \mathcal{P}_{++}A_0\Big(v(1-b_1)(1-b_2)\Big) + \mathcal{P}_{+-}A_0\Big(v(1-b_1)(1+b_2)\Big) \\ & +\mathcal{P}_{-+}A_0\Big(v(1+b_1)(1-b_2)\Big) + \mathcal{P}_{--}A_0\Big(v(1+b_1)(1+b_2)\Big) \\ \equiv \ & \mathcal{P}_{++}V_{--} + \mathcal{P}_{+-}V_{-+} + \mathcal{P}_{-+}V_{+-} + \mathcal{P}_{--}V_{++} \ . \end{aligned} \tag{4.5}$$

Reexpressing the projectors $\mathcal{P}_{++}$, etc., in terms of the matrices $\gamma_{a\ell}$ ($a = 1,2$), one obtains:

$$\begin{aligned} \overline{A}_0 = \ & \frac{1}{4}[V_{--} + V_{-+} + V_{+-} + V_{++}] + \frac{1}{4}[V_{--} + V_{-+} - V_{+-} - V_{++}]i\gamma_{1\ell} \\ & +\frac{1}{4}[V_{--} - V_{-+} + V_{+-} - V_{++}]i\gamma_{2\ell} + \frac{1}{4}[V_{--} - V_{-+} - V_{+-} + V_{--}]i^2\gamma_{1\ell}\gamma_{2\ell} \ . \end{aligned} \tag{4.6}$$

This expression should be identified with the combination $(A+iB_1\gamma_{1\ell}+iB_2\gamma_{2\ell}-C\gamma_{1\ell}\gamma_{2\ell})$ appearing in Eq. (3.16) on the left or the right of $\gamma_{1L}\gamma_{2L}\gamma_1.\gamma_2$ (the substitution (4.4) being done symmetrically with respect to this operator). We then obtain the identifications:

$$\begin{aligned}
A &= \frac{1}{4}[V_{--}+V_{-+}+V_{+-}+V_{++}]\ , \\
B_1 &= \frac{1}{4}[V_{--}+V_{-+}-V_{+-}-V_{++}]\ , \\
B_2 &= \frac{1}{4}[V_{--}-V_{-+}+V_{+-}-V_{++}]\ , \\
C &= \frac{1}{4}[V_{--}-V_{-+}-V_{+-}+V_{++}]\ ,
\end{aligned} \tag{4.7}$$

and from Eqs. (3.24:

$$\begin{aligned}
\alpha_+ &= V_{--}+V_{++}\ , \qquad \alpha_- = V_{-+}+V_{+-}\ , \\
\beta_+ &= V_{--}-V_{++}\ , \qquad \beta_- = V_{-+}-V_{+-}\ .
\end{aligned} \tag{4.8}$$

For the particular case of Todorov's potential (2.15), the expressions of the potentials V_{++}, etc., are:

$$\begin{aligned}
V_{--} &= \frac{1}{4}\ln\Big(1+4v(1-b_1)(1-b_2)\Big)\ , \\
V_{-+} &= \frac{1}{4}\ln\Big(1+4v(1-b_1)(1+b_2)\Big)\ , \\
V_{+-} &= \frac{1}{4}\ln\Big(1+4v(1+b_1)(1-b_2)\Big)\ , \\
V_{++} &= \frac{1}{4}\ln\Big(1+4v(1+b_1)(1+b_2)\Big)\ .
\end{aligned} \tag{4.9}$$

5 Reduction to a final eigenvalue equation

In the absence of anomalous magnetic moments, the wave equations (3.7) or (3.10) can be reduced to a single Pauli-Schrödinger type equation for the component ψ_3 or ψ_{B3} [22]. A similar reduction can also be undertaken here; however, due to the complexity of the new terms in the effective potential [Eq. (3.22)], the reduction process is not as straightforward as before. The reason is that the components ψ_i or ψ_{Bi} $(i = 1, \ldots, 4)$ [Eqs. (3.12)-(3.13)] do no longer have, in the general case, simple characterizations with the quantum numbers ℓ (orbital angular momentum) and s (total spin). For instance, in the absence of anomalous magnetic moments, the component ψ_3 can be classified according to the quantum numbers $\ell = j \pm 1$, $s = 1$ (j being the total angular momentum) and $\ell = j$. In the present case, this simple property is lost and such a classification will concern combinations of ψ_3 and ψ_2.

It turns out that the most convenient representation where the reduction process can be achieved is the "anti-Breit" representation defined with the wave function transformations $\chi = e^{V_B}\psi = e^{2V_B}\psi_B$ [cf. Eq. (3.9)]. In this case the reduced wave function is a tractable combination of χ_3 and χ_2. We shall not, however, present here the reduced wave equation in the general cases of quantum numbers, the corresponding expression being still lengthy, but rather shall content ourselves with the simplest case of the $j = 0$ quantum number, corresponding to the two sectors of pseudoscalar and scalar states. These are also the most sensitive sectors involved in zero-mass bound state problems in strong coupling regimes.

Actually, for these sectors, the Breit representation (3.9) is the simplest one and it is sufficient to project Eq. (3.10) along the components ψ_{Bi} $(i = 1, \ldots, 4)$ [Eq. (3.13)]. In these sectors, the operators w_{12}, $(W_{1S} + W_{2S}).\hat{x}^T$ and $(W_{1S} + W_{2S}).p^T$ have the following quantum numbers: $w_{12} = 1$, $(W_{1S} + W_{2S}).\hat{x}^T = 0$, $(W_{1S} + W_{2S}).p^T = 0$. Equation (3.10), together with Eq. (3.22), then yields the following four coupled equations:

$$
\begin{aligned}
P_L e^{\alpha_+\gamma_+}(f_{++} + g_{++})\psi_{B1} - (m_1 - m_2)\psi_{B2} + (\frac{2}{\hbar P_L})(W_{1S} - W_{2S}).p^T\psi_{B3}& \\
- \frac{i}{2}P_L(\frac{2}{\hbar P_L})(W_{1S} - W_{2S}).\hat{x}^T e^{\alpha_+\gamma_-}h_{+-}\psi_{B4} &= 0\ , \\
P_L e^{\alpha_-\gamma_-}(f_{--} - g_{--})\psi_{B2} - (m_1 - m_2)\psi_{B1}& \\
+ \frac{i}{2}P_L(\frac{2}{\hbar P_L})(W_{1S} - W_{2S}).\hat{x}^T e^{\alpha_-\gamma_+}h_{-+}\psi_{B3} &= 0\ ,
\end{aligned}
$$

$$
\begin{aligned}
P_L e^{\alpha_-\gamma_+}(f_{-+}+g_{-+})\psi_{B3} - M\psi_{B4} + (\frac{2}{\hbar P_L})(W_{1S}-W_{2S}).p^T\psi_{B1} \\
-\frac{i}{2}P_L(\frac{2}{\hbar P_L})(W_{1S}-W_{2S}).\hat{x}^T e^{\alpha_-\gamma_-}h_{--}\psi_{B2} &= 0 \ , \\
P_L e^{\alpha_+\gamma_-}(f_{+-}-g_{+-})\psi_{B4} - M\psi_{B3} \\
+\frac{i}{2}P_L(\frac{2}{\hbar P_L})(W_{1S}-W_{2S}).\hat{x}^T e^{\alpha_+\gamma_+}h_{++}\psi_{B1} &= 0 \ .
\end{aligned}
\tag{5.1}
$$

These equations allow the elimination of the three components ψ_{B1}, ψ_{B2} and ψ_{B4} in terms of ψ_{B3}, which is a surviving component in the nonrelativistic limit. Defining

$$
e^{2h_{+-,-+}} = 1 - \frac{(m_1-m_2)^2}{P^2} e^{-(\alpha_-\gamma_+ + \alpha_+\gamma_-)} \frac{(f_{+-}-g_{+-})}{(f_{-+}-g_{-+})} \ , \tag{5.2}
$$

$$
e^{-2u} = e^{-\alpha_+\gamma_-} (f_{+-}-g_{+-})e^{-2h_{+-,-+}} \ , \tag{5.3}
$$

and making the wave function transformation

$$
\psi_{B3} \ = \ e^{u}\phi_3 \ , \tag{5.4}
$$

one obtains the following eigenvalue equation for ϕ_3, written, for simplicity, in the c.m. frame:

$$
\begin{aligned}
\Big\{ \ & \frac{P^2}{4}\frac{e^{\alpha_-\gamma_+ + \alpha_+\gamma_-}}{(f_{+-}-g_{+-})(f_{-+}-g_{-+})} - \frac{M^2}{4}\frac{1}{(f_{+-}-g_{+-})^2} - \mathbf{p}^2 \\
& -\frac{(m_1^2-m_2^2)^2}{4M^2}\frac{1}{(f_{-+}-g_{-+})^2} + \frac{(m_1^2-m_2^2)^2}{4P^2}\frac{(1+h_{+-}^2)e^{-(\alpha_-\gamma_+ + \alpha_+\gamma_-)}}{(f_{+-}-g_{+-})(f_{-+}-g_{-+})} \\
& -4\hbar^2\mathbf{x}^2 \left[u' + \frac{M}{4\hbar r}\frac{h_{+-}}{(f_{+-}-g_{+-})} - \frac{(m_1-m_2)}{4\hbar r}\frac{h_{-+}}{(f_{-+}-g_{-+})} \right]^2 \\
& +(6\hbar^2-4\mathbf{S}^2) \left[u' + \frac{M}{4\hbar r}\frac{h_{+-}}{(f_{+-}-g_{+-})} - \frac{(m_1-m_2)}{4\hbar r}\frac{h_{-+}}{(f_{-+}-g_{-+})} \right] \\
& +4\hbar^2\mathbf{x}^2 \left[u'' + \frac{M}{4\hbar}\left(\frac{h_{+-}}{r(f_{+-}-g_{+-})}\right)' - \frac{(m_1-m_2)}{4\hbar}\left(\frac{h_{-+}}{r(f_{-+}-g_{-+})}\right)' \ \right] \Big\} \ \phi_3 \ = \ 0 \ .
\end{aligned}
\tag{5.5}
$$

Here, the prime designates derivation with respect to r^2 $(=\mathbf{x}^2)$:

$$
f' \ \equiv \ \frac{\partial f}{\partial r^2} \ ; \tag{5.6}
$$

$\mathbf{S}$ is the total spin operator, $\mathbf{S} = \mathbf{s_1} + \mathbf{s_2}$, $\mathbf{S}^2 = 2\hbar^2 s$, $s = 0, 1$; the other operators and functions are defined in Eqs. (3.23)-(3.24), (5.2)-(5.3). The eigenvalues of the matrices

$\gamma_\pm$ in the sectors with $j = 0$ are the following:

$$\begin{aligned} j\ &=\ 0\ ,\ \ell\ =\ 0\ ,\ s\ =\ 0\ : \quad \gamma_+\ =\ 3\ , \quad \gamma_-\ =\ -1\ ; \\ j\ &=\ 0\ ,\ \ell\ =\ 1\ ,\ s\ =\ 1\ : \quad \gamma_+\ =\ -1\ , \quad \gamma_-\ =\ 3\ . \end{aligned} \tag{5.7}$$

The sector with $\ell = 0$, $s = 0$ corresponds to the pseudoscalar states, while the sector with $\ell = 1$, $s = 1$ corresponds to the scalar states.

6 Properties of the eigenvalue equation

We study in this section two aspects of the eigenvalue equation (5.5) concerning, first, its short-distance singularities, and, second, its nonrelativistic limit.

6.1 Short-distance singularities

The question that arises here is whether the presence of the anomalous magnetic moments has any influence on the short-distance singularities of the effective potentials present in the eigenvalue equation. It was already clear from the expressions of the substitutions (2.11) that the form factors $b_a(r)$ $(a = 1, 2)$ should be bounded in modulus by 1 in order not to destabilize at finite distances the bound state system. A detailed analysis of the eigenvalue equation is however necessary to reach a more complete understanding of the role of the form factors near the origin. We shall limit our study to the case of Todorov's potential (2.15).

For a matter of comparison, we rewrite Eq. (5.5) in the case when the anomalous magnetic moments are absent. Here, we have $B_1 = B_2 = C = 0$, $A = A_0$, $\alpha_+ = \alpha_- = 2A_0$, $\beta_+ = \beta_- = 0$. Denoting $h \equiv h_{+-,-+}$ [Eq. (5.2)] in this case, Eq. (5.5) becomes [22]:

$$\begin{aligned}\Big\{ & \frac{P^2}{4}e^{8A_0} - \frac{M^2}{4}e^{4A_0} - \frac{(m_1^2 - m_2^2)^2}{4M^2}e^{4A_0} + \frac{(m_1^2 - m_2^2)^2}{4P^2} \\ & -\mathbf{p}^2 - 4\hbar^2\mathbf{x}^2 h'^2 + 6\hbar^2 h' + 4\hbar^2\mathbf{x}^2 h'' \\ & -4\mathbf{S}^2 \Big[(2A_0' + h')(1 + 4\mathbf{x}^2 A_0') - (A_0' + 2\mathbf{x}^2 A_0'') \Big] \Big\} \phi_3 \ = \ 0 \ .\end{aligned} \tag{6.1}$$

It is sufficient to study the short-distance singularity problem in the equal-mass case ($m_1 = m_2$, $h = 1$) and in the ground state sector ($\ell = 0, s = 0$). The dominant singularity comes from the term $\frac{P^2}{4}e^{8A_0}$, which, according to the expression (2.15) of A_0, yields the attractive potential α^2/r^2. This term is at the origin of the fall-to-the-center phenomenon with a critical value of α equal to $\frac{1}{2}$ [24].

The above analysis can be repeated with Eq. (5.5). In the equal-mass case, one has $b_1 = b_2 = b$ [Eqs. (2.11)] and the expressions of the various potentials [Eqs. (3.23)-(3.24), (4.2), (4.7)-(4.9)] become:

$$
\begin{aligned}
\alpha_+ &= \frac{1}{4}\ln\left[\Big(1+4v(1-b)^2\Big)\Big(1+4v(1+b)^2\Big)\right] , \\
\alpha_- &= \frac{1}{2}\ln\Big(1+4v(1-b^2)\Big) , \\
\beta_+ &= \frac{1}{4}\ln\left[\frac{1+4v(1-b)^2}{1+4v(1+b)^2}\right] , \qquad \beta_- \ = \ 0 , \\
f_{+-}-g_{+-} &= \cosh\sqrt{\alpha_+^2+\beta_+^2\gamma_-^2} - \frac{\alpha_+}{\sqrt{\alpha_+^2+\beta_+^2\gamma_-^2}}\sinh\sqrt{\alpha_+^2+\beta_+^2\gamma_-^2} , \\
f_{-+}-g_{-+} &= e^{-\alpha_-} .
\end{aligned} \tag{6.2}
$$

Their behaviors near the origin are:

$$
\begin{aligned}
\alpha_+ \simeq \alpha_- \ &\simeq \ \frac{1}{2}\ln(\frac{1}{P_L r}) , \\
\beta_+ &\simeq \frac{1}{4}\ln\left(\frac{1-b}{1+b}\right)^2 , \qquad 0 \le b(0) < 1 , \\
f_{+-}-g_{+-} &\simeq e^{-\alpha_+}\left(1-\frac{\beta_+^2\gamma_-^2}{2\alpha_+^2}\right) + \frac{\beta_+^2\gamma_-^2}{4\alpha_+^2}e^{\alpha_+} \\
&\simeq (P_L r)^{1/2} + \beta_+^2\gamma_-^2\left(\ln(\frac{1}{P_L r})\right)^{-2}(P_L r)^{-1/2} , \\
f_{-+}-g_{-+} &\simeq (P_L r)^{1/2} .
\end{aligned} \tag{6.3}
$$

The behavior of $(f_{+-}-g_{+-})$ near the origin crucially depends on that of $b(r)$. If $b(0) \neq 0$, then $\beta_+(0) \neq 0$ and hence $(f_{+-}-g_{+-})$ essentially behaves as $r^{-1/2}$. The first term in Eq. (5.5) has therefore a behavior of the type r^{-1}, contrary to the behavior of the type r^{-2} obtained in the absence of anomalous magnetic moments. Therefore, a non-vanishing of the form factors $b(r)$ at the origin drastically changes the singularity of the effective potential at the origin. Also in this case, for $s=0$, the function u [Eq. (5.3) behaves as $-\alpha_+$ and the combination $-4\hbar^2\mathbf{x}^2u'^2 + 6\hbar^2 u' + 4\hbar^2\mathbf{x}^2 u''$ of Eq. (5.5) has a behavior close to $\hbar^2/(4r^2)$, which was absent in the initial case. This singularity is independent of the value of the coupling constant α and is located at the critical point. This would mean that the system, even for small values of α, would face strong attractive singularities, which are not observed experimentally.

The above study suggests that the form factors $b(r)$ should vanish at the origin, in order not to drastically modify the situation found in the absence of anomalous magnetic moments. A smooth contribution of the anomalous magnetic moments would require

that the second term in the right-hand side of the equation of $(f_{+-} - g_{+-})$, Eq. (6.3), be nondominant in front of the first. This implies that β_+, and hence b, vanish at least as rapidly as $r^{1/2}$ at the origin. (Also, in this case, the function u vanishes at the origin.)

A parametrization of $b_a(r)$, corresponding to a vanishing at the origin as r, is obtained with the following choice of the functions $c_a(r)$ of Eqs. (2.12):

$$c_a(r) \;=\; d_a + f_a \frac{\hbar\alpha}{2\pi m_a r} \qquad (a = 1, 2) \;, \tag{6.4}$$

with d_a and f_a constants, $d_a > 1$, $f_a > 0$.

6.2 Nonrelativistic limit

When the magnetic moment form factors $b(r)$ are smooth functions, then for values of the coupling constant α of the order of 1/2 (the critical value), their effects can still be estimated perturbatively, their order of magnitude being fixed by α/π. An even cruder estimate is obtained by the nonrelativistic limit, which allows us to have easily an idea of the signs of the energy shifts. We shall assume that α is sufficiently small to also justify a perturbative-nonrelativistic treatment of the Coulomb potential v [Eq. (4.2)] appearing in expressions concerning the anomalous magnetic moments.

We treat the form factors $b_a(r)$ to first order.Among the effective potentials (3.23)-(3.24) and (4.8)-(4.9), only $\beta_\pm$ and the h's are first order quantities in b_a. In this approximation, the latter are given by the perturbation theory result:

$$b_a(r) \;\simeq\; \frac{\alpha}{2\pi}\frac{1}{2m_a r} \qquad (a = 1, 2) \;. \tag{6.5}$$

One finds for the effective potentials:

$$\begin{aligned} h_{+-} \;&\simeq\; \beta_+\gamma_- \;, \qquad h_{-+} \;\simeq\; \beta_-\gamma_+ \;, \\ \beta_+ \;&\simeq\; -2v(b_1 + b_2) \;, \qquad \beta_- \;\simeq\; -2v(b_1 - b_2) \;. \end{aligned} \tag{6.6}$$

Let the first-order perturbation due to the anomalous magnetic moments appearing in Eq. (5.5) be represented by $-\delta V$. We have:

$$\begin{aligned} -\delta V = (6\hbar^2 - 4\mathbf{S}^2) & \left[\frac{M}{4\hbar r}h_{+-} - \frac{(m_1 - m_2)}{4\hbar r}h_{-+}\right] \\ & +4\hbar^2\mathbf{x}^2 \left[\frac{M}{4\hbar r}h_{+-} - \frac{(m_1 - m_2)}{4\hbar r}h_{-+}\right]' \end{aligned}$$

$$
\begin{aligned}
= \ & \nabla^2 \int^{\mathbf{x}^2} dr^2 \Big[\frac{M}{4\hbar r} h_{+-} - \frac{(m_1 - m_2)}{4\hbar r} h_{-+}\Big] \\
& -4\mathbf{S}^2 \left[\frac{M}{4\hbar r} - \frac{(m_1 - m_2)}{4\hbar r} h_{-+}\right] ,
\end{aligned} \tag{6.7}
$$

where ∇^2 is the laplacian operator. Using the expressions of the h's and b's [Eqs. (6.5)-(6.6)] we also have:

$$
\begin{aligned}
\frac{M}{4\hbar r} h_{+-} - \frac{(m_1 - m_2)}{4\hbar r} h_{-+} &= \frac{1}{4\hbar r}\Big[- M 2v(b_1 + b_2)\gamma_- + (m_1 - m_2) 2v(b_1 - b_2)\gamma_+\Big] \\
&= -\frac{1}{16\pi\hbar}\frac{1}{m_1 m_2 M}\frac{\alpha^2}{r^3}\Big(M^2\gamma_- + (m_1 - m_2)^2\gamma_+\Big) .
\end{aligned} \tag{6.8}
$$

Then the corresponding perturbation in the nonrelativistic hamiltonian, designated by δV_{NR}, is related to δV with the relation [22]:

$$
\delta V \ = \ \frac{2m_1 m_2}{M}\delta V_{NR} . \tag{6.9}
$$

Introducing the total spin quantum number s $(= 0, 1)$, we obtain:

$$
\begin{aligned}
\delta V_{NR} = \ & \frac{\alpha^2}{4\hbar}\frac{1}{m_1^2 m_2^2}\delta^3(\mathbf{x})[M^2\gamma_- + (m_1 - m_2)^2\gamma_+] \\
& -\frac{\alpha^2}{4\pi\hbar}\frac{1}{m_1^2 m_2^2}\frac{s}{r^3}[M^2\gamma_- + (m_1 - m_2)^2\gamma_+] .
\end{aligned} \tag{6.10}
$$

The first term contributes to the sector with $s = 0$, $\ell = 0$, for which $\gamma_+ = 3$, $\gamma_- = -1$ [Eqs. (5.7)], while the second one to the sector with $s = 1$, $\ell = 1$, for which $\gamma_+ = -1$, $\gamma_- = 3$. The energy shift then becomes:

$$
\begin{aligned}
\delta E = \ & \frac{\alpha^5}{2\pi}\frac{m_1 m_2}{M^3}(m_1^2 + m_2^2 - 4m_1 m_2)\frac{\delta_{\ell 0}\delta_{s0}}{n_\ell^3} \\
& -\frac{\alpha^5}{6\pi}\frac{m_1 m_2}{M^3}(m_1^2 + m_2^2 + 4m_1 m_2)\frac{\delta_{\ell 1}\delta_{s1}}{n_\ell^3} ,
\end{aligned} \tag{6.11}
$$

with $n_\ell = \ell + n' + 1$, $n' = 0, 1, \ldots$.

While the sign of the energy shift is negative for the sector with $s = 1$, $\ell = 1$, it depends on the ratio m_1/m_2 for the sector with $s = 0$, $\ell = 0$. For the particular case of equal masses, $m_1 = m_2$, the energy shift for the latter sector is negative and equal to the energy shift of the sector with $s = 1$, $\ell = 1$.

In the infinite mass limit, $m_2 \to \infty$, the problem reduces to that of a spin-$\frac{1}{2}$ particle with anomalous magnetic moment placed in an external static Coulomb field. In this

case, the sector with $s = 0$, $\ell = 0$ tends to the new sector with $j = \frac{1}{2}$, $\ell = 0$ and the sector with $s = 1$, $\ell = 1$ to the sector with $j = \frac{1}{2}$, $\ell = 1$. Equations (6.10) and (6.11) become:

$$\begin{aligned} \delta V_{NR} &= \frac{\alpha^2}{2m_1^2}\delta^3(\mathbf{x}) - \frac{\alpha^2}{2m_1^2}\delta_{\ell 1}\frac{1}{r^3} \ , \\ \delta E &= m_1\frac{\alpha^5}{2\pi}\frac{\delta_{\ell 0}\delta_{j1/2}}{n_\ell^3} - m_1\frac{\alpha^5}{6\pi}\frac{\delta_{\ell 1}\delta_{j1/2}}{n_\ell^3} \ . \end{aligned} \tag{6.12}$$

They agree, as they should, with the corresponding formulas obtained directly from the Dirac equation [25]. (Comparisons of theoretical predictions involving anomalous magnetic moments with experimental data can be found in Ref. [26].)

7 Summary and concluding remarks

Using a matrix substitution rule, applied to the electric charge and deduced from the lowest order contribution of the vertex correction in QED, we introduced in local form the anomalous magnetic moments at each vertex of the higher order terms of the constraint theory fermion-antifermion interaction potential. Since the latter already has a local form in three-dimensional x-space, determined from summation of infra-red leading terms of multiphoton exchange diagrams, the new potential that arises also has a similar locality property and is calculated by a resummation of the corresponding series after the incorporation of the anomalous magnetic moments into the vertices.

Focusing our attention to the sectors of pseudoscalar and scalar states ($j = 0$), the corresponding wave equations were reduced to a single eigenvalue equation. The requirement that the short-distance singularities of the effective potential should not be drastically enhanced by the presence of the anomalous magnetic moments imposed on the accompanying form factors the condition of a sufficiently rapid vanishing at the origin (faster than $r^{1/2}$). It is expected that when this condition is realized, then the incorporation of the anomalous magnetic moments, even in the strong coupling regime, should not introduce qualitative or destabilizing changes in the properties of the fermion-antifermion bound state system.

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